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Dynamics of Generalized Assisted Inflation

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Abstract

We study the dynamics of multiple scalar fields and a barotropic fluid in an FLRW-universe. The scalar potential is a sum of exponentials. All critical points are constructed and these include scaling and de Sitter solutions. A stability analysis of the critical points is performed for generalized assisted inflation, which is an extension of assisted inflation where the fields mutually interact. Effects in generalized assisted inflation which differ from assisted inflation are emphasized. One such a difference is that an (inflationary) attractor can exist if some of the exponential terms in the potential are negative.

1 Introduction

Scalar fields can violate the strong energy condition, which is a necessary condition for inflation and present-day acceleration. An alternative way to violate the strong energy condition is pure vacuum energy, that is, a positive cosmological constant Λ. Observations do not exclude either possibility, but scalar fields provide more freedom to evade typical fine-tuning problems posed by a cosmological constant such as the smallness of Λ or the cosmic coincidence problem. On the other hand it is argued that scalar fields generate new problems not posed by a cosmological constant [1].

In this paper we investigate scalar field models and we confine ourselves to models where the scalar fields have a canonical kinetic term and the potential is a sum of exponential terms where the exponent is a linear combination of the scalars. The reason for this choice is twofold. First of all, these models allow to find exact solutions, which correspond to critical points in an autonomous system [1]. The second reason is that exponential potentials arise in models motivated by string theory such as supergravities obtained from dimensional reduction (see for instance [3–7] and references therein), descriptions of brane interactions [8–10], nonperturbative effects and the effective description of string gas cosmology (see for instance [11]).

The simplest example is the single-field model with one positive exponential potential, which was shown to have a stable powerlaw inflationary solution when the potential is sufficiently flat [12]. Later on this model was extended by including a barotropic fluid, negative exponentials, spatial curvature and combinations thereof [13–17]. In all cases it is possible to rewrite the equations of motion as an autonomous system. Some critical points correspond to so-called scaling solutions. Scaling solutions are defined as solutions where different constituents of the universe are in coexistence such that the ratios of their energy densities stays constant during evolution.

Since supergravities typically contain many scalars we are interested in the extension of these models to multiple scalars. In reference [18] the authors consider a potential that is a sum of exponentials with each exponential containing a different scalar. The main result is that if each separate exponential is too steep to drive inflation the system can still have an inflationary attractor provided the number of fields is sufficiently large. This effect is known as assisted inflation. In reference [19] this system is extended to include a barotropic fluid and spatial curvature.

In assisted inflation there is no mutual interaction (cross-coupling) between the scalars. To understand the effects of mutually interacting scalars reference [20] investigates a model with one exponential term containing multiple scalars. But as we will argue this theory can be rewritten as a theory without cross-coupling. Mutual interaction of the scalar fields can only be present when there are more exponentials terms in the potential.

As shown in [21] multiple exponential potentials are naturally divided into two classes:

1. The minimal number of scalar fields which can occur in the potential equals the number of exponentials.

[1] For a review on dynamical systems in cosmology see [2].

2. The minimal number of scalar fields which can occur in the potential is less than the number of exponentials.

A more precise definition can be found in section 2. The first class can have scaling solutions and the second class can have scaling and de Sitter solutions.

Here we shall mostly be interested in class 1. This class is termed generalized assisted inflation in [22]. Generalized assisted inflation includes assisted inflation as a special case where there is no cross-coupling. In reference [22] some exact solutions of generalized assisted inflation were given, which were later shown to correspond to a subset of all possible critical points [21].

In contrast to assisted inflation, generalized assisted inflation is poorly investigated. Therefore we extend the results of [21, 22] in two ways. First, by constructing all possible critical points of the model containing the most general multiple exponential potential with a barotropic fluid and spatial curvature. The second and more important extension is a stability analysis of the critical points in generalized assisted inflation.

The paper is organized as follows. In section 2 the equations of motion are given in a flat FLRW-background and are then rewritten as an autonomous system. In section 3 we construct all critical points for the most general multiple exponential potential. Section 4 contains a full stability analysis of all critical points in generalized assisted inflation. In section 5 the model is generalized by the inclusion of nonzero spatial curvature and finally in section 6 we summarize and conclude. In the appendices we prove some results.

2 The model

The scalars are denoted by ϕ_i and are assembled in an N-vector ϕ. The model is defined by the following action

$$S = \int d^4x\,\sqrt{-g}\Big[\frac{1}{2\kappa^2}\mathcal{R} - \tfrac{1}{2}\langle\partial_\mu\phi\,,\,\partial^\mu\phi\rangle - V(\phi)\Big] + S_M\,, \tag{1}$$

where $\kappa^2 = 8\pi G$ with G Newton's constant and $\langle\partial_\mu\phi\,,\,\partial^\mu\phi\rangle$ is shorthand for $\sum_{i=1}^N \partial_\mu\phi_i\partial^\mu\phi_i$. The potential $V(\phi)$ is a sum of M exponential terms

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[-\kappa\,\langle\alpha_a\,,\,\phi\rangle], \tag{2}$$

where $\langle\alpha_a\,,\,\phi\rangle = \sum_{i=1}^N \alpha_{ai}\phi_i$. There are M vectors α_a with N components α_{ai}. The indices $i, j, \ldots$ run from 1 to N and denote the components of the vectors ϕ and α_a. The indices $a, b, \ldots$ run from 1 to M and label the different vectors α_a and the constants Λ_a. S_M is the action for a barotropic fluid with density ρ and pressure P and as equation of state $P = (\gamma - 1)\rho$. It is assumed that the barotropic fluid respects the strong energy condition, which means that $2/3 < \gamma < 2$.

In this paper we make use of linear field redefinitions. If the scalars transform linearly as $\phi \longrightarrow \phi' = S\phi$, where S is an element of GL$(\mathbb{R}, N)$ then the vectors α_a transform in the dual representation $\alpha_a \longrightarrow \alpha'_a = S^{-T}\alpha_a$. This can be seen from the definition of α'_a

$$\langle \alpha_a \,,\, \phi \rangle \equiv \langle \alpha'_a \,,\, \phi' \rangle \,. \tag{3}$$

Field redefinitions that shift the scalar fields leave the α_a invariant, but change the Λ_a.

From the action we can deduce some properties of this system by looking at transformations in scalar space. The kinetic term is invariant under constant shifts and $O(N)$-rotations of the scalars. These transformations map the multiple exponential potential to another multiple exponential potential but with different Λ_a and α_a. Such redefinitions do not alter the physics, they just rewrite the equations. Therefore qualitative features only depend on $O(N)$-invariant combinations of the α_a-vectors (for example $\langle \alpha_a \,,\, \alpha_b \rangle$). By shifting the scalars we can always rescale R of the Λ_a to be ± 1, where R is the number of independent α-vectors.

The Ansatz for the metric is that of a flat FLRW-universe and accordingly the scalars can only depend on the cosmic time τ

$$ds^2 = -d\tau^2 + a(\tau)^2 d\vec{x}^{\,2} \,, \quad \phi_i = \phi_i(\tau) \,. \tag{4}$$

In section 5 the analysis is extended to spatially curved FLRW-universes. The equations of motion are

$$H^2 = \tfrac{\kappa^2}{3}\Big[\tfrac{1}{2}\langle \dot\phi \,,\, \dot\phi \rangle + V(\phi) + \rho\Big] \,, \tag{5}$$

$$\ddot\phi_i + 3H\dot\phi_i + \partial_i V(\phi) = 0 \,, \tag{6}$$

$$\dot\rho + 3\gamma H \rho = 0 \,, \tag{7}$$

where $H = \dot a / a$ is the Hubble constant. The dot denotes differentiation with respect to τ. The equations (5-7) are referred to as the Friedmann equation, the Klein–Gordon equation and the continuity equation, respectively.

One can rewrite equations (5-7) as an autonomous system. We define the following dimensionless variables

$$X_i = \frac{\kappa \dot\phi_i}{\sqrt{6}\, H}, \quad Y_a = \frac{\kappa^2}{3\, H^2}\, \Lambda_a \exp[-\kappa \, \langle \alpha_a \,,\, \phi \rangle], \quad \Omega = \frac{\kappa^2 \rho}{3H^2} \,. \tag{8}$$

If we write $X^2 = \sum_i X_i^2$ and $Y = \sum_a Y_a$ the equations of motion become

$$\Omega + X^2 + Y - 1 = 0 \,, \tag{9}$$

$$X'_i = 3X_i\Big(-1 + X^2 + \tfrac{\gamma}{2}\Omega\Big) + \sqrt{\tfrac{3}{2}} \sum_a \alpha_{ai} Y_a \,, \tag{10}$$

$$Y'_a = Y_a\Big(-\sqrt{6}\, \langle \alpha_a \,,\, X \rangle + 6X^2 + 3\gamma\Omega\Big) \,, \tag{11}$$

$$\Omega' = 3\Omega\Big(2X^2 + \gamma(\Omega - 1)\Big) \,, \tag{12}$$

where the prime denotes differentiation with respect to $\ln(a)$ [2].

It can be shown that if $\Omega + X^2 + Y - 1 = 0$ initially then equations (10-12) guarantee that it is satisfied at all times. Hence, given the correct initial conditions the dynamics is described by equations (10-12).

The number of linearly independent vectors α_a is denoted by R. If $R < N$ one can rotate the scalars such that $\phi_{R+1}, \dots, \phi_N$ no longer appear in the potential ($\alpha_{ai} = 0$ for $i > R$). These scalars are then said to be decoupled or free.

In the next section we will construct all critical point solutions of the autonomous system (10-12). For that purpose it is useful to consider the matrix A

$$A_{ab} = \langle \alpha_a \, , \, \alpha_b \rangle \, . \tag{13}$$

The models are divided into two classes. The first class is defined by $R = M$ and the second class by $R < M$. Algebraically the two differ in the following way

$$1.\;\; R = M \quad \Longleftrightarrow \quad \det A > 0 \, , \tag{14}$$

$$2.\;\; R < M \quad \Longleftrightarrow \quad \det A = 0. \tag{15}$$

The first possibility, where A is invertible, is called generalized assisted inflation.

Assisted inflation is the subclass where A is diagonal. This implies that in assisted inflation the α_a are perpendicular to each other and that one can choose an orthonormal basis in which $\alpha_{ai} = \alpha_a \, \delta_{ai}$. In that basis the potential becomes

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[-\kappa \, \alpha_a \, \phi_a] \, . \tag{16}$$

It is this particular form of the potential that is referred to as assisted inflation in the literature. We want to emphasize that the latter definition is basis-dependent. Potentials different from (16) but with a diagonal matrix A can be brought to the form (16) through an $O(N)$-rotation of the scalar fields. An $O(N)$-invariant definition of assisted inflation is that A is a diagonal matrix.

In any system with multiple fields but a single exponential such as studied in [20], the matrix A is trivially diagonal. One can perform a rotation on the scalars such that only one scalar appears in the potential and all the others are decoupled. In order to have a system whose scalars are mutually interacting one needs at least two exponential terms both containing more than one scalar.

[2]The use of $\ln(a)$ as a time coordinate fails when $\dot{a} = 0$. This is no problem for studying critical points and their stability because in the neighborhood around a critical point there always exists a region where the coordinate is well defined. In reference [16] an explicit example is given where $\dot{a}$ becomes zero at some point.

3 The critical points

Critical points are defined as solutions of the autonomous system for which $\Omega' = X_i' = Y_a' = 0$. From the acceleration equation,

$$\dot{H} = -\tfrac{\kappa^2}{2}\Big[\langle\dot{\phi}\,,\,\dot{\phi}\rangle + \gamma\rho\Big]\,, \tag{17}$$

it follows that in a critical point $\dot{H}/H^2$ is constant. If the constant differs from zero we put $\dot{H}/H^2 \equiv -1/p$ and then the scale factor becomes

$$a(\tau) = a_0\,\Big(\frac{\tau}{\tau_0}\Big)^p\,. \tag{18}$$

In terms of the dimensionless variables p can be expressed as

$$p = \frac{1}{3(X^2 + \frac{\gamma}{2}\Omega)}\,. \tag{19}$$

When $\dot{H}/H^2 = 0$ the scale factor is

$$a(\tau) = a_0\,e^{H\tau}\,, \tag{20}$$

and space-time is de Sitter.

The requirement that $Y_a' = 0$ can be satisfied in two ways as can be seen from

$$Y_a' = Y_a\Big(-\sqrt{6}\,\,\langle\alpha_a\,,\,X\rangle + 6X^2 + 3\gamma\Omega\Big)\,. \tag{21}$$

Either $Y_a = 0$ or the second factor on the right hand side equals zero. If we put $Y_a = 0$ by hand and then solve for the X_i and the remaining Y_a, the critical point is called a *nonproper critical point*. If we put the second factor to zero by hand and then solve for X_i and Y_a, the critical point is called a *proper critical point*. The name nonproper is given since a critical point with some $Y_a = 0$ has ∞-valued scalar fields. Therefore these critical points are no proper solutions to the equations of motion, they are asymptotic descriptions of solutions. The proper critical points generically have nonzero Y_a and therefore are proper solutions to the equations of motion. But in some cases one finds that $Y_a = 0$ for proper critical points, although one did not put those Y_a to zero by hand.

Regardless of whether critical points are proper solutions to the equations of motion, they are all equally important in providing information about the orbits. That is, they are either repellers, attractors or saddle points.

3.1 Generalized assisted inflation

Proper critical points

First we construct the proper critical points by proceeding as explained in [21] and find

$$Y_a = 2\frac{3p-1}{3p^2}\sum_b [A^{-1}]_{ab}\,, \quad X_i = \frac{1}{p}\sqrt{\tfrac{2}{3}}\sum_{ab}\alpha_{ai}[A^{-1}]_{ab}\,, \quad \Omega = 1 - \frac{2}{p}\sum_{ab}[A^{-1}]_{ab}\,. \tag{22}$$

The value of p is found by combining equations $(9, 19, 22)$. One gets

$$p^2 - p\,(2\sum_{ab}[A^{-1}]_{ab} + \tfrac{2}{3\gamma}) + \tfrac{4}{3\gamma}\sum_{ab}[A^{-1}]_{ab} = 0\,. \tag{23}$$

The two possible solutions are

$$p_\phi = 2\sum_{ab}[A^{-1}]_{ab} \quad \text{and} \quad p_{\phi+\rho} = \frac{2}{3\gamma}\,. \tag{24}$$

The first solution has vanishing barotropic fluid ($\Omega = 0$) and is called *the scalar-dominated solution*. The second solution is a *matter-scaling solution* [15]; the ratio of the fluid density Ω to the scalar field density $\Omega_\phi \equiv \kappa^2\rho_\phi/3H^2$ (with $\rho_\phi \equiv \frac{1}{2}\langle\dot\phi\,,\,\dot\phi\rangle + V$) is finite and constant. For the matter-scaling solution the scalar field mimics the fluid resulting in $p = \frac{2}{3\gamma}$, which is the same powerlaw for a universe filled with only the barotropic fluid.

In terms of the scalar fields the solutions read

$$\phi_i = \frac{\sqrt{6}X_i\,p}{\kappa}\ln|\tau| + c_i\,. \tag{25}$$

Nonproper critical points

To construct nonproper critical points we put a certain subset of $Y_{\bar a}$ equal to zero by hand and then solve for the X_i and the remaining Y_a. We denote the elements of the subset of Y_a that were put to zero with barred indices $\bar a$, so $Y_{\bar a} = 0$.

The solutions are still given by equations (22) and (24) but now the indices a, b, c run over the subset of nonzero Y_a and the matrix A has to be replaced by the matrix $A(\bar a, \bar b, \ldots)$, which is the submatrix of A formed by deleting the $\bar a, \bar b, \ldots$ columns and rows of A.

Similar to the proper critical points, the nonproper critical points are divided into matter-scaling solutions and scalar-dominated solutions. If all $Y_a = 0$ the Friedmann equation (9) gives that $X^2 + \Omega = 1$. From (12) we notice that either $\Omega = 0$ or $X^2 = 0$. For the first case where $\Omega = 0$, the solution is called a *kinetic-dominated solution* and $p = 1/3$. For the second case where $\Omega = 1$, the solution corresponds to a universe only filled with a barotropic fluid and is termed the *fluid-dominated solution* ($p_\rho = 2/3\gamma$).

The maximal number of critical points can be seen to be

$$\text{Max}\,\sharp\,\text{critical points} = 2 \times 2^M. \tag{26}$$

The reason is that there are 2^M different ways of putting Y_a equal to zero. For each combination there are two possibilities: when not all Y_a equal zero there is the scalar-dominated and the matter-scaling solution and when all $Y_a = 0$ there is the fluid-dominated solution and the kinetic-dominated solution.

For the scalar-dominated solutions the nonproper critical points correspond to universes with an expansion that is always slower than that of the proper critical point. This is due to the following lemma.

Consider an invertible matrix A with as entries innerproducts of vectors $A_{ab} = \langle \alpha_a \, , \, \alpha_b \rangle$. Define $A(\bar{a}, \bar{b}, \ldots)$ as before, then we have the following inequality

$$\sum_{ab} [A^{-1}]_{ab} \geq \sum_{cd} [A(\bar{a}, \bar{b}, \ldots)^{-1}]_{cd} \, . \tag{27}$$

For diagonal A this property is obvious. The proof for general A can be found in appendix A.

Existency conditions

The critical points constructed above do not always exist. For instance, it is clear from the definition of the Y_a-variables that they must have the same sign as the Λ_a. So, if the solution for Y_a does not respect that, it has to be discarded[3]. Of all critical points two always exist, the kinetic-dominated and fluid-dominated solution. For the other critical points it depends on whether they are matter-scaling or scalar-dominated solutions.

For the existence of a matter-scaling solution there is an extra condition coming from $\Omega \geq 0$. For the matter-scaling solution the conditions $\Omega \geq 0$ and $\mathrm{sgn}(Y_a) = \mathrm{sgn}(\Lambda_a)$ become (22)

$$\sum_{ab} [A^{-1}]_{ab} \leq \frac{1}{3\gamma} \, , \quad \sum_b A^{-1}_{ab} \gtrless 0 \quad \text{for} \quad \Lambda_a \gtrless 0 \, . \tag{28}$$

This shows for instance that for a single negative exponential potential there is no matter-scaling solution [16].

If the critical point is a scalar-dominated solution then existence is guaranteed if

$$(\frac{1}{p_\phi} - 3) \sum_b [A^{-1}]_{ab} \gtrless 0 \quad \text{for} \quad \Lambda_a \gtrless 0 \, . \tag{29}$$

For the nonproper critical points the same equations hold by changing A to $A(\bar{a}, \bar{b} \ldots)$.

To conclude, in generalized assisted inflation we distinguish four kinds of critical points: the (non)proper scalar-dominated solutions, the (non)proper matter-scaling solutions, the fluid-dominated solution and the kinetic-dominated solution. The kinetic and fluid-dominated solutions always exist and for the matter-scaling and the scalar-dominated solutions the existency conditions are given in (28) and (29), respectively.

3.2 The $R < M$ case

This section is brief since details can be found in [21].

An important difference with the previous case where $R = M$ is that these potentials can have stationary points $\partial_i V(\phi) = 0$ which correspond to de Sitter solutions ($a \sim e^{\tau}$)

[3] In [4] it is shown that critical points that violate the existency conditions can still play a rol in understanding the late time behavior of general solutions.

when $V > 0$. Since $R < M$ the α_a-vectors are linearly dependent and we can always choose an independent set of R vectors α_a with $a = 1 \ldots R$. The remaining vectors α_b ($b = R \ldots M$) can be expressed as linear combinations of that independent set

$$\alpha_b = \sum_{a=1}^{R} c_{ba}\alpha_a \,. \tag{30}$$

Concerning the proper critical points one has to distinguish between the case of potentials with *affine* coupling[4] and those without. By affine coupling we mean the situation where we can find a set of independent α_a's such that the coefficients c_{ba} obey

$$\sum_a c_{ba} = 1 \;\text{ for all }\; b \,. \tag{31}$$

These couplings are for instance found in group manifold reductions of pure gravity [4]. If the couplings are affine then the proper critical points correspond to powerlaw solutions.

Since the inverse of the matrix A does not exist when $R < M$ the formulas for the solutions differ. We introduce the symmetric matrix B

$$B_{ij} = \sum_{a=1}^{M} \alpha_{ai}\alpha_{aj} \,. \tag{32}$$

If we assume that a basis rotation is performed whereby the decoupled scalars are removed from the potentials we can restrict the index i to run from 1 to R. The decoupled scalars obey $X_R = \ldots = X_N = 0$. With this restriction on the index i the matrix B becomes an $R \times R$ invertible matrix. The powerlaw solutions when the coupling is affine is given by

$$X_i = \frac{\sqrt{2/3}}{p} \sum_{ja} B^{-1}_{ij}\alpha_{aj} \,. \tag{33}$$

For p there are again two possibilities

$$p_\phi = 2\sum_i \Big(\sum_{ja} [B^{-1}]_{ij}\alpha_{aj}\Big)^2 \quad \text{and} \quad p_{\phi+\rho} = \tfrac{2}{3\gamma} \,. \tag{34}$$

Therefore there exist proper scaling solutions (matter-scaling and scalar-dominated) when the number of exponential terms exceeds the number of scalars, but only if the coupling is affine. When $R < M$ and the coupling is not affine there are no proper scaling solutions, but there can still exist nonproper scaling solutions (which are referred to as approximate scaling solutions in reference [5]).

When the couplings are not affine the only possible proper critical point is a de Sitter universe.

[4]In [21] we called this coupling a convex coupling. Later we found that the name affine is more appropriate.

For the construction of the nonproper critical points a subset of the Y_a must equal zero. Depending on the subset that is put to zero, the resulting equations are that of the $R = M$ or $R < M$-case. So in general nonproper critical points in the $R < M$ case can be de Sitter and powerlaw.

The existence of the critical points is more complicated than for generalized assisted inflation since the Y_a-variables are no longer independent variables but are nonlinear dependent.

4 Stability analysis

To determine the stability of a critical point one linearizes the equations of motion around the critical point. The matrix that appears in the linearized autonomous equations is called the stability matrix. Stability of the critical point requires that the real parts of the eigenvalues of the stability matrix are negative.

We will not discuss the stability of critical points for models with $R < M$ but only for $R = M$. The reason is the difficulty that arises from the dependence of the Y_a. It is a hard problem to write down the independent perturbations around a critical point. There exist exceptions where the perturbations are independent when $R = M - 1$, see [23] for an example where $R = 1$, $M = 2$.

Assume (X_i, Y_a, Ω) is a critical point and consider a perturbation $(X_i+\delta_i, Y_a+\delta_a, \Omega+\delta_\Omega)$. If we substitute the perturbations in equations (10) and (11) and keep the linear terms we find

$$\delta_i' = (\tfrac{1}{p} - 3)\delta_i + 3(2-\gamma)\sum_j X_i X_j \delta_j - \sum_a (\tfrac{3}{2}\gamma X_i \delta_a - \sqrt{\tfrac{3}{2}}\alpha_{ai}\delta_a)\,, \tag{35}$$

$$\delta_a' = Y_a \sum_j \Big(6(2-\gamma)X_j - \sqrt{6}\alpha_{aj}\Big)\delta_j + \delta_a\Big(-\sqrt{6}\langle\alpha_a\,,\,X\rangle + \tfrac{2}{p} - 3\gamma Y_a\Big)\,. \tag{36}$$

The δ_Ω perturbations are obtained by varying the Friedmann equation (9). Under this assumption it is not an independent perturbation. The equations above can be written as a matrix equation $\delta' = M\,\delta$. From (35, 36) one can read off the values of the stability matrix M

$$\begin{aligned}
M_{ij} &= [\tfrac{1}{p} - 3]\mathbb{1}_{ij} + 3(2-\gamma)X_i X_j\,,\\
M_{ib} &= -\tfrac{3\gamma}{2}X_i + \sqrt{\tfrac{3}{2}}\alpha_{bi}\,,\\
M_{aj} &= 6(2-\gamma)Y_a X_j - \sqrt{6}\alpha_{aj}Y_a\,,\\
M_{ab} &= \mathbb{1}_{ab}\Big(-\sqrt{6}\langle\alpha_a\,,\,X\rangle + \tfrac{2}{p}\Big) - 3\gamma Y_a\,.
\end{aligned} \tag{37}$$

We now discuss the decoupled scalars $\phi_{R+1}, \ldots, \phi_N$. Since $\alpha_{aR+1} = \ldots = \alpha_{aN} = 0$ it follows from equation (10) that either $X_{R+1} = \ldots = X_N = 0$ or some $X_{i>R} \neq 0$. The latter only happens for the kinetic-dominated solution $(X^2 = 1, \Omega = 0)$. The effect of free scalars at the kinetic-dominated solution is discussed in [4]. For all critical points except the kinetic-dominated solution M has only diagonal components in the directions $i > R$:

$$M_{ib} = M_{aj} = 0 \ , \ M_{ij} = (\tfrac{1}{p} - 3)\delta_{ij} \quad \text{for} \quad i > R \,. \tag{38}$$

The decoupled scalars decouple in the stability-analysis and do not introduce instabilities as long as $p > 1/3$.

We first investigate the eigenvalues of the stability matrix at the nonproper critical points and then at the proper critical points.

4.1 Stability of nonproper critical points

The submatrix M_{ij} is diagonalizable with an $O(R)$-transformation that rotates the scalars. The eigenvalues are

$$E_1 = [\tfrac{1}{p} - 3] \quad , \quad E_2 = [\tfrac{1}{p} - 3] + 3(2-\gamma)X^2 \,. \tag{39}$$

The multiplicity of E_1 is $R-1$ and the multiplicity of E_2 is 1. This is sufficient to understand the stability of the critical points for which all $Y_a = 0$; the kinetic-dominated and the fluid-dominated solution. For these two solutions we then diagonalize the M_{ij}-submatrix via an $O(R)$-rotation and we end up with a matrix M which is upper triangular. The eigenvalues can then be read off from the diagonal. For the fluid-dominated solution the eigenvalues are $3(\gamma/2 - 1) < 0$ and $3\gamma > 0$. Therefore the fluid-dominated solution is always unstable. For the kinetic-dominated solution the eigenvalues are $3(2-\gamma) > 0$ and $-\sqrt{6}\langle\alpha_a \, , \, X\rangle + 2/p$, the first eigenvalue is always positive and hence the kinetic-dominated solution is always unstable.

Assume a certain nonproper critical point has one Y_a equal to zero and all other Y_a nonzero. After renumbering we can choose it to be Y_1. The Y_1-perturbation decouples from all the other perturbations. The reason is that if $Y_1 = 0$ the $a = 1$ row of M only contains its diagonal element $M_{a=1\,a=1}$ and therefore the determinant of $M - \lambda$ becomes

$$\det(M - \lambda) = (M_{a=1\,a=1} - \lambda)\det(\tilde{M} - \lambda) \,, \tag{40}$$

where $\tilde{M}$ denotes the matrix obtained after deleting the $a = 1$ row and $a = 1$ column from M. The matrix $\tilde{M}$ is the same as the stability matrix of the proper critical point belonging to the new system that one obtains by removing the $\Lambda_1 \exp[-\kappa\langle\alpha_1 \, , \, \phi\rangle]$-term from the potential.

This means that the stability of the nonproper critical point with $Y_1 = 0$ and $Y_{a>1} \neq 0$ boils down to checking the sign of $M_{a=1\,a=1}$ and to study the stability of the proper critical point of the truncated system $\tilde{M}$. The stability of proper critical points is left for the next section and we now focus on the sign of $M_{a=1\,a=1}$.

Stability requires

$$M_{a=1\,a=1} = -\sqrt{6}\langle\alpha_1 \, , \, X\rangle + 2/p < 0 \,. \tag{41}$$

In appendix A we prove that this expression is equivalent to

$$\sum_b [A^{-1}]_{1b} < 0\,. \tag{42}$$

For nonproper critical points with more Y_a's equal to zero, $Y_1 = Y_2 = \ldots = Y_C = 0$ we find C conditions

$$\sum_{b=1}^{R-C+1} [A(2,\ldots,C)]^{-1}_{1b} < 0 \;\;, \ldots \;, \;\sum_{b=1}^{R-C+1} [A(1,\ldots,C-1)]^{-1}_{1b} < 0\,. \tag{43}$$

If $\sum_b [A^{-1}]_{ab} > 0$ for all $a = 1\ldots M$ then equation (43) cannot be satisfied (see appendix A) and there does not exist a stable truncation of the Y_a. In the case of a diagonal A matrix this property is obvious, hence we find that in assisted inflation nonproper critical points are always unstable.

4.2 Stability of proper critical points

The characteristic polynomial

If we consider perturbations around a proper critical point a simplification occurs in the stability matrix. The M_{ab} submatrix reduces to $M_{ab} = -3\gamma Y_a$, since now

$$\langle \alpha_a\,,\, X\rangle = \tfrac{2}{\sqrt{6}p}\,. \tag{44}$$

The eigenvalues of M are found by solving the characteristic polynomial $\det(M - \lambda\mathbb{1}) = 0$. The computation is simplified using an LU-decomposition of $M - \lambda\mathbb{1}$. The characteristic polynomial becomes

$$\det(M_{ij} - \lambda\mathbb{1}_{ij})\det(Z_{ab}) = 0\,, \tag{45}$$

where we defined the matrix Z

$$Z_{ab} = M_{ab} - \lambda\mathbb{1}_{ab} - M_a{}^i[(M - \lambda\mathbb{1})^{-1}]_i{}^j M_{jb}\,. \tag{46}$$

The LU-decomposition is not valid when λ is an eigenvalue of M_{ij} ($E_{1,2}$). Solutions to $\det(Z) = 0$ are eigenvalues of M if they differ from $E_{1,2}$. With this LU-decomposition one has to check separately whether $E_{1,2}$ are eigenvalues of M.

The matrix Z_{ab} contains only contracted i-indices and is therefore $O(R)$-invariant. We can choose a specific orthonormal basis to compute it. The easiest choice is that basis in which M_{ij} is diagonal, where we have

$$M_a{}^i[(M - \lambda\mathbb{1})^{-1}]_i{}^j M_{jb} = \frac{1}{E_1 - \lambda} M_a{}^i M_{ib} + \left(\frac{1}{E_2 - \lambda} - \frac{1}{E_1 - \lambda}\right) M_{aR} M_{Rb}\,. \tag{47}$$

This expression contains X_R and α_{aR} in the specific basis. One can determine them explicitly since by definition of the new basis $X = (0,\ldots,0,\sqrt{X^2})$. Using this and the fact that expression (44) is $O(R)$-invariant we find that $\alpha_{aR} = 2/p\sqrt{6X^2}$ and hence

$$Z_{ab} = cY_a + \frac{3}{E_1 - \lambda} A_{ab} Y_a - \lambda\delta_{ab}\,, \tag{48}$$

with c the following expression[5]

$$c = -3\gamma + \frac{1}{E_2 - \lambda}\Big[9(2-\gamma)\gamma X^2 + \frac{3\gamma - 12}{p} + \frac{2}{p^2 X^2}\Big] - \frac{2}{p^2 X^2 (E_1 - \lambda)}\,. \tag{49}$$

If one adds all rows to the first row and then subtracts the first column from all the other columns the new matrix Z' one obtains has the same determinant as Z and is given by

$$Z' = \begin{bmatrix} cY + \frac{6p-2}{p^2}\frac{1}{E_1-\lambda} - \lambda & 0 \\ cY_a + \frac{3}{E_1-\lambda}A_{a1} & \frac{3}{E_1-\lambda}(A_{ab>1} - A_{a1}) - \delta_{ab}\lambda \end{bmatrix}. \tag{50}$$

This shows that the determinant factorizes in a c-dependent and a c-independent polynomial

$$\det(Z') = \Big(cY + \frac{6p-2}{p^2}\frac{1}{E_1-\lambda} - \lambda\Big)\det\Big(\frac{3}{E_1-\lambda}(A_{ab} - A_{a1}) - \delta_{ab}\lambda\Big) \text{ with } a, b > 1\,. \tag{51}$$

The factorization of the characteristic polynomial in a c-dependent and a c-independent part allows us to divide the perturbations into fluid perturbations and scalar field perturbations. The fluid perturbations are those that give rise to the eigenvalues that come from the c-dependent factor whereas the scalar field perturbations belong to those eigenvalues that come from the c-independent factor.

Fluid perturbations

If we put the first factor of (51) to zero we get the following eigenvalues for the scalar-dominated solution

$$\lambda_1 = E_1\,, \quad \lambda_2 = \frac{2}{p} - 3\gamma\,, \tag{52}$$

and for the matter-scaling solution

$$\lambda_\pm = \lambda = \frac{3}{4}(2-\gamma)\Big[-1 \pm \sqrt{1 - 8\gamma\frac{1 - 3\gamma\sum_{ab}[A^{-1}]_{ab}}{2-\gamma}}\Big]. \tag{53}$$

These eigenvalues have a simple interpretation. Stability of the scalar-dominated solution requires

$$\lambda_2 < 0 \quad \Longleftrightarrow \quad p > 2/3\gamma \quad \Longleftrightarrow \quad p_\phi > p_{\phi+\rho}. \tag{54}$$

For stability it does not matter whether E_1 is an eigenvalues or not, because its implication for stability ($p > 1/3$) is less strong then the implications coming from λ_2 ($p > 2/3\gamma$) when we assume that $2/3 < \gamma < 2$.

Stability of the matter-scaling solution requires

$$\mathrm{Re}\lambda_\pm < 0 \quad \Longleftrightarrow \quad p > 2\sum_{ab}[A^{-1}]_{ab} \quad \Longleftrightarrow \quad p_{\phi+\rho} > p_\phi. \tag{55}$$

[5]For the special case of $R = 1$ the formulae can be taken over if one replaces E_1 by E_2 everywhere.

The existency condition for the matter-scaling solution (28) implies stability (55) which in turn implies instability of the scalar-dominated solution. In [24] it is proven that for lagrangians that allow for scaling solutions but without cross-coupling, the scalar-dominated solution is unstable when the matter-scaling solution is stable.

Scalar field perturbations

The remaining eigenvalues follow from the smaller polynomial

$$\det\Big(\frac{3}{E_1-\lambda}(A_{ab}-A_{a1})-\delta_{ab}\lambda\Big)=0 \quad \text{with} \quad a,b>1\,. \tag{56}$$

Determining the solutions of (56) for arbitrary A is a hard problem and we will use another technique to determine stability. This technique does not use the (linearized) autonomous equations and was developed in the context of assisted inflation [18] and elaborated upon in [25]. Below we will extend this to the case of generalized assisted inflation.

The technique is based upon making a basis rotation in scalar space such that one direction corresponds to the critical point solution and the other directions are constructed perpendicular to it. Then the idea is to check whether the critical point direction is a minimum of the potential with respect to the other direction, if so, the critical point is an attractor.

In order to understand which basis rotation to make notice that the proper critical points are given by

$$\phi_i(\tau)=\frac{\sqrt{6}X_i p}{\kappa}\ln|\tau|+c_i\,, \tag{57}$$

irrespective of whether it is a matter-scaling or a scalar-dominated solution. Combinations of the form

$$\frac{\phi_i}{X_i}-\frac{\phi_j}{X_j}\,, \tag{58}$$

are constant in time at the critical point. Therefore we define the following new scalar combinations

$$\phi'_i=N_i[\frac{\phi_i}{X_i}-\frac{\phi_{i+1}}{X_{i+1}}] \qquad \text{for} \quad i<R\,, \tag{59}$$

$$\phi'_R=N_R[\frac{\phi_1}{X_2X_3\ldots X_R}+\frac{\phi_2}{X_3X_4\ldots X_RX_1}+\ldots+\frac{\phi_R}{X_1X_2\ldots X_{R-1}}]\,, \tag{60}$$

with N_i and N_R normalization constants given by

$$N_i=\Big[X_i^{-2}+X_{i+1}^{-2}\Big]^{-1/2}\,, \quad N_R=X^{-1}\,\Pi_{i=1}^{R}X_i\,. \tag{61}$$

The directions $\phi'_{i<R}$ are all perpendicular to the direction ϕ'_R, but they are not perpendicular among themselves, hence this is not an orthogonal field redefinition. To establish orthogonality we perform a Gram–Schmidt procedure on the $\phi'_{i<R}$ directions. The Gram–Schmidt procedure takes linear combinations of the $\phi'_{i<R}$ to obtain new directions $\phi''_{i<R}$

that are mutually perpendicular. It follows from the construction that the $\phi''_{i<R}$ are also constant at the critical point. We define $\phi''_R \equiv \phi'_R$.

The transition of the ϕ_i to the ϕ''_i is done orthogonally and therefore the system still has a canonical kinetic term and a multiple exponential potential

$$S = \int d^4x\,\sqrt{-g}\Big[\frac{1}{2\kappa^2}\mathcal{R} - \tfrac{1}{2}\langle\partial_\mu\phi''\,,\,\partial^\mu\phi''\rangle - \sum_{a=1}^{M}\Lambda_a \exp[-\kappa\,\langle\alpha''_a\,,\,\phi''\rangle]\Big]\,. \tag{62}$$

We focus on the appearance of ϕ''_R in the potential, so we have to find what α''_{aR} is. The scalars are redefined by an orthogonal transformation, $\phi'' = O\phi$. The α-vectors transform as dual vectors $\alpha''_a = [O^{-1}]^T\,\alpha_a$ which simplifies to $\alpha''_a = O\,\alpha_a$ for orthogonal transformations. In components this means that

$$\alpha''_{ai} = O_i^{\ j}\alpha_{aj}\,. \tag{63}$$

To find α''_{aR} it then suffices to construct $O_R^{\ i}$ for all i.

Although we do not know the matrix O since we did not specify the Gram–Schmidt procedure, we can find $O_R^{\ i}$ from (60) since $\phi'_R = \phi''_R$

$$O_R^{\ i} = \frac{X_i}{X}\,. \tag{64}$$

Thus in the potential the scalar ϕ''_R appears in the a-th exponent as

$$\alpha''_{aR}\phi''_R \;=\; X^{-1}\langle X\,,\,\alpha_a\rangle\;\phi''_R \;=\; \frac{2\phi''_R}{\sqrt{6}\,p\,X}\,. \tag{65}$$

The product $\alpha''_{aR}\phi''_R$ is independent of a, so all exponentials have the same coupling to ϕ''_R and we can rewrite the potential as

$$V(\phi'') = e^{\frac{2\phi''_R}{\sqrt{6}\,p\,X}}\,W(\phi''_1,\ldots,\phi''_{R-1})\,,\quad W(\phi''_1,\ldots,\phi''_{R-1}) = \sum_{a=1}^{M}\Lambda_a\,e^{-\kappa\,\langle\alpha''_a\,,\phi''\rangle}\,, \tag{66}$$

and $\langle\alpha''_a\,,\,\phi''\rangle$ does not contain ϕ''_R;

$$\langle\alpha''_a\,,\,\phi''\rangle = \sum_{i=1}^{R-1}\alpha''_{ai}\phi''_i\,. \tag{67}$$

The fact that we were able to rewrite the potential like (66) extends the results of [25] to the case of generalized assisted inflation.

The function $W(\phi''_1,\ldots,\phi''_{R-1})$ in (66) contains $R-1$ scalar fields and has R exponential terms. These functions can have a stationary point ($\partial V = 0$) and this stationary point is unique. Moreover this stationary point corresponds to the critical point, the solution for which the fields $\phi''_1, \ldots, \phi''_{R-1}$ take constant values. This is proven by considering the Klein–Gordon equations in the new basis

$$\ddot{\phi}''_i + 3H\dot{\phi}''_i + \partial_i(e^{\frac{2\phi''_R}{\sqrt{6}\,p\,X}}\ W) = 0\,. \tag{68}$$

Since for the critical point $\phi''_1, \ldots, \phi''_{R-1}$ are constant we see that

$$\partial_{i<R}(W(\phi''_1, \ldots, \phi''_{R-1})) = 0\,. \tag{69}$$

The critical point is an attractor if and only if this stationary point is a minimum of W. Stability requires that the matrix $\partial_i\partial_j W(\phi''_1, \ldots, \phi''_{R-1})$ (for $i, j = 1\ldots, R-1$) is positive definite. If all $\Lambda_a > 0$ then $\partial_i\partial_j W$ is positive definite since $\partial_i\partial_j W = \alpha^T\eta\alpha$ with η a diagonal matrix with positive entries, $\eta_{aa} = \Lambda_a\exp[-\kappa\langle\alpha''_a\,,\,\phi''\rangle] > 0$. Hence if the proper critical point of a potential with all $\Lambda_a > 0$ exists, it is stable. When some $\Lambda_a < 0$ we have proven in appendix B that the matrix $\partial_i\partial_j W = \alpha^T\eta\alpha$ is not positive definite if $p > 1/3$. Hence potentials with some negative Λ_a cannot have stable *proper* critical points with $p > 1/3$.

4.3 Summary of the stability analysis

We discuss qualitatively the results first for assisted inflation and then for generalized assisted inflation. We explain how to find the stable critical points (attractors) by eliminating all unstable critical points.

Assisted inflation

At the end of section 4.1 we found that in assisted inflation all nonproper critical points are unstable. So if an attractor is present it is either the proper matter-scaling or the proper scalar-dominated solution.

Because of the existency conditions for critical points with diagonal A $(28, 29)$, the proper critical point only exist when all $\Lambda_a > 0$[6]. This implies that for assisted inflation models with some negative Λ_a there is no attractor, all critical points are unstable.

In subsection 4.2 we analyzed the eigenvalues coming from the c-dependent factor of the characteristic polynomial $\det(Z) = 0$. The stability conditions are $p_\phi > p_{\phi+\rho}$ for the scalar-dominated solution and $p_\phi < p_{\phi+\rho}$ for the matter-scaling solution. This means that the stable proper critical point is the one that represents the universe with the fastest expansion of the two. The nonproper critical points are all unstable and have smaller p-values, which implies that the attractor is the critical point with the fastest expansion of *all* the critical points. Furthermore, when the matter-scaling solution exists, it is stable and the scalar-dominated solution is unstable.

In sum, assisted inflation models have an attractor when all exponential terms are positive and furthermore the attractor is unique and corresponds to the critical point with

[6]We assume that $p > 1/3$, since scalar-dominated solutions with $p_\phi > 1/3$ are unstable because one can proof that $E_1 = 1/p - 3$ is always an eigenvalue of the stability matrix for assisted inflation.

the fastest expansion. Thus in assisted inflation the universe dynamically favors a fast expansion!

A fast expansion is favored because of the 'assisting behavior' of the scalar fields. Each scalar field gives a positive contribution to the Hubble friction and thus more scalar fields lead to more friction and more friction leads to slow rolling fields. Therefore one says that the scalars assist each other in helping the universe to expand. The assisting behavior is proven for all multi-field lagrangians that allow for scaling solutions *and* have no cross-coupling of the scalars [24].

Generalized assisted inflation

We separately discuss the cases where all terms in the potential are positive and where some are negative.

1. $\forall\ \Lambda_a > 0$. If a proper critical point exists, the scalar field perturbations are stable since all $\Lambda_a > 0$. As in assisted inflation the fluid perturbations imply that the stable proper critical point is the one that represents the universe with the fastest powerlaw, either the matter-scaling (if $p_{\phi+\rho} > p_\phi$) or the scalar-dominated solution (if $p_\phi > p_{\rho+\phi}$). Since the proper critical point exists we have $\sum_b [A^{-1}]_{ab} > 0$ for all b [7], which implies that truncations are unstable (see discussion below (43)). Hence the nonproper critical points are unstable and the attractor is unique. The cases analyzed in [7] are of this type.

   If there exists no proper critical point, attractors have to be nonproper. To find stable nonproper critical points, the stability of the truncation (42,43) is the only condition that has to be fulfilled. Therefore nonproper critical points can be stable when the proper critical point does not exist [8]. It is not guaranteed that there is a unique stable truncation nor if a stable truncation is possible at all. Hence the number of attractors is case dependent and can be zero, one or higher.

2. $\exists\ \Lambda_a < 0$. Whereas critical points with negative Y_a do not exist in assisted inflation, they can exist in generalized assisted inflation but are unstable as we now explain. If $Y_{n_1}, \ldots, Y_{n_q}$ are negative at a critical point $\wp$ than so are $\Lambda_{n_1}, \ldots, \Lambda_{n_q}$ in order for $\wp$ to exist. If $\wp$ is proper we proved at the end of section 4.2 that it must be unstable. If $\wp$ is nonproper i.e. $Y_{s_1} = \ldots = Y_{s_p} = 0$ then we showed in section 4.1 that the $Y_{s_1}, \ldots, Y_{s_p}$-perturbations decouple from all other perturbations. The latter perturbations are the same perturbations as that of a proper critical point $\tilde{\wp}$ of a lagrangian where $\Lambda_{s_1}, \ldots, \Lambda_{s_p}$ are put to zero. This truncated lagrangian contains the negative $\Lambda_{n_1}, \ldots, \Lambda_{n_q}$ hence $\tilde{\wp}$ is unstable. Since the perturbations around $\wp$ contain the perturbations around $\tilde{\wp}$, $\wp$ must also be unstable. This proves that proper and nonproper critical points with negative Y_a are unstable.

[7] We use that $p_\phi > 1/3$ for scalar-dominated solutions of positive potentials.

[8] But the kinetic- and fluid-dominated solution can never be stable as we showed in section 4.1.

Assume $\Lambda_1, \ldots, \Lambda_n < 0$ and all other $\Lambda_a > 0$. If some of the $Y_1, \ldots, Y_n$ differ from zero they are negative in order for the critical point to exist. Therefore we have to truncate at least all $Y_1, \ldots, Y_n$ to find a stable critical point. If such truncations are unstable, there is no attractor. Again, it is not guaranteed that there is a unique stable truncation nor if a stable truncation is possible at all.

5 The effects of spatial curvature

We show that for critical points with $k = 0$ curvature perturbations do not affect the eigenvalues found in section 4. Instead curvature perturbations lead to an additional eigenvalue.

The metric Ansatz is

$$ds^2 = -d\tau^2 + a(\tau)^2\Big(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\Big), \tag{70}$$

where $d\Omega^2$ is the metric on the unit 2-sphere and $k = 0, \pm 1$ is the normalized curvature. The equations of motion are

$$H^2 = \tfrac{\kappa^2}{3}\Big[\tfrac{1}{2}\langle\dot{\phi}\,,\,\dot{\phi}\rangle + V(\phi) + \rho\Big] - \frac{k}{a^2}\,, \tag{71}$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \partial_i V(\phi) = 0\,, \tag{72}$$

$$\dot{\rho} + 3\gamma H\rho = 0\,. \tag{73}$$

Using the same dimensionless variables as in section 2, the equations of motion become

$$\Omega + X^2 + Y - 1 = \frac{k}{\dot{a}^2}\,, \tag{74}$$

$$X_i' = X_i(Q-2) + \sqrt{\tfrac{3}{2}}\sum_a \alpha_{ai} Y_a\,, \tag{75}$$

$$Y_a' = Y_a\Big(-\sqrt{6}\,\langle\alpha_a\,,\,X\rangle + 2(Q+1)\Big)\,, \tag{76}$$

$$\Omega' = \Omega\Big(2Q - (3\gamma - 2)\Big)\,. \tag{77}$$

where in the last line a new quantity Q, the deceleration parameter, is introduced. It is defined by

$$Q \equiv -\frac{a\ddot{a}}{\dot{a}^2} = 2X^2 - Y + \Omega(\frac{3\gamma}{2} - 1)\,. \tag{78}$$

For powerlaw solutions $a \sim \tau^p$ the deceleration parameter is simply $Q = \frac{1-p}{p}$.

One might wonder if the system is still autonomous due to the $\frac{k}{\dot{a}^2}$-term. Using equations (75-77) one shows that

$$\big(X^2 + Y + \Omega - 1\big)' = 2Q\big(X^2 + Y + \Omega - 1\big)\,. \tag{79}$$

The solution to this equation is

$$|X^2 + Y + \Omega - 1| = \frac{C}{\dot{a}^2}, \tag{80}$$

with $C \geq 0$. If equation (74) is satisfied at the instant $\tau = \tau_0$, it is satisfied at all times. We conclude that the evolution never changes the value of k and that the dynamics is still described by the autonomous system (75-77).

The critical points can be divided into critical points with $k = 0$ and critical points with $k \neq 0$. The critical points with $k = 0$ are the same as the ones discussed in section 3. The critical points with $k \neq 0$ necessarily have $p = 1$.

Curvature perturbations

We define the function $f(X, Y, \Omega)$ by

$$f(X, Y, \Omega) = X^2 + Y + \Omega - 1. \tag{81}$$

The function f provides us with information about the spatial curvature, for example $\mathrm{sgn} f = k$. Consider perturbations $X_i + \delta_i$, $Y_a + \delta_a$ and $\Omega + \delta_\Omega$. They induce a perturbation δf which we will call a curvature perturbation. It satisfies the following differential equation

$$(\delta f)' = 2Q\delta f + 2f\delta Q. \tag{82}$$

In section 4 ($f = 0$) we considered perturbations that satisfy $\delta f = 0$. In order to study the stability of the critical point against curvature perturbations we need to treat δ_i, δ_a and δ_Ω as independent perturbations. The stability matrix M can be calculated in exactly the same manner as in section 4.2 with the only difference that the resulting matrix has one extra row and one extra column. The entries are

$$\begin{aligned}
&M_{\Omega\Omega} = 2Q + (3\gamma - 2)(\Omega - 1)\,, && M_{\Omega j} = 8\Omega X_j\,,\\
&M_{a\Omega} = (3\gamma - 2)Y_a\,, && M_{i\Omega} = (\tfrac{3}{2}\gamma - 1)X_i\,,\\
&M_{aj} = -\sqrt{6}Y_a\alpha_{aj} + 8Y_aX_j\,, && M_{\Omega b} = -2\Omega\,,\\
&M_{ij} = (Q - 2)\mathbb{1}_{ij} + 4X_iX_j\,, && M_{ib} = -X_i + \sqrt{\tfrac{3}{2}}\alpha_{bi}\,,\\
&M_{ab} = -2Y_a + \big(2(Q+1) - \sqrt{6}\textstyle\sum_i \alpha_{ai}X_i\big)\mathbb{1}_{ab}\,. &&
\end{aligned} \tag{83}$$

Performing an LU-decomposition of the matrix $M - \lambda\mathbb{1}$ one shows that

$$\det(M - \lambda\mathbb{1}) = (M_{\Omega\Omega} - \lambda)\det(M_{ij} - \lambda\mathbb{1}_{ij})\det(C_{ij})\det(Z_{ab}) = 0, \tag{84}$$

where C_{ij} and Z_{ab} are given by

$$C_{ij} = \mathbb{1}_{ij} - \sum_k (M_{ik} - \lambda\mathbb{1}_{ik})^{-1}\frac{M_{k\Omega}M_{\Omega j}}{M_{\Omega\Omega} - \lambda}\,, \tag{85}$$

$$Z_{ab} = M_{ab} - \lambda\mathbb{1}_{ab} - \frac{M_{a\Omega}M_{\Omega b}}{M_{\Omega\Omega} - \lambda} - \sum_{l,k,n} C^{-1}_{nk}(M_{kl} - \lambda\mathbb{1}_{kl})^{-1}\big(M_{an} - \frac{M_{a\Omega}M_{\Omega n}}{M_{\Omega\Omega} - \lambda}\big)\big(M_{lb} - \frac{M_{l\Omega}M_{\Omega b}}{M_{\Omega\Omega} - \lambda}\big).$$

This decomposition is valid if

$$\lambda \neq M_{\Omega\Omega}\,, \quad \lambda \neq E_1\,, \quad \lambda \neq E_2\,, \quad \det(C_{ij}) \neq 0\,, \tag{86}$$

where $E_1 = Q - 2$, $E_2 = Q - 2 + 4X^2$ are the eigenvalues of M_{ij}. Making use of the basis in which $X = (0, \ldots, 0, \sqrt{X^2})$ as explained in section 4 we find

$$\det(C_{ij}) = 1 - 4\Omega \frac{3\gamma - 2}{E_2 - \lambda} \frac{X^2}{M_{\Omega\Omega} - \lambda}\,, \tag{87}$$

$$Z_{ab} = cY_a + \frac{3}{E_1 - \lambda} A_{ab} Y_a - \lambda \mathbb{1}_{ab}\,, \tag{88}$$

where the constant c is

$$\begin{aligned} c = &-2 + 2\Omega \frac{3\gamma - 2}{M_{\Omega\Omega} - \lambda} - \frac{2}{E_1 - \lambda} \frac{(Q+1)^2}{X^2} - \frac{1}{\det C_{ij}} \frac{2}{E_2 - \lambda} \Big[5(Q+1) - 4X^2 \\ &- \frac{(Q+1)^2}{X^2} - 5(Q+1)\Omega \frac{3\gamma - 2}{M_{\Omega\Omega} - \lambda} + 8\Omega X^2 \frac{3\gamma - 2}{M_{\Omega\Omega} - \lambda} - 4X^2\Omega^2 \left(\frac{3\gamma - 2}{M_{\Omega\Omega} - \lambda} \right)^2 \Big]\,. \end{aligned} \tag{89}$$

The matrix Z is of the same form as in section 4.2 but with a different constant c. In section 4.2 it was shown that for proper critical points the determinant of Z_{ab} factorizes in a c-dependent and a c-independent polynomial. This result also applies to proper critical points with $k \neq 0$. Therefore the only eigenvalues that change are those that are related to c. These eigenvalues are determined by

$$\lambda = cY + \frac{6p - 2}{p^2} \frac{1}{E_1 - \lambda}\,. \tag{90}$$

This proves that the effects of curvature only manifest themselves in a subset of the eigenvalues of the stability matrix. In this sense the effects of curvature decouple from the scalar dynamics.

Next we discuss the effect of curvature perturbations for the scalar-dominated and scaling solutions with $k = 0$ as well as the stability of the new critical points with $k \neq 0$. In both cases we only consider proper critical points. For the case of nonproper critical points the arguments of section 4.1 apply here as well.

Stability of critical points with $k = 0$

For the scalar-dominated solution we find the eigenvalue

$$\lambda_1 = M_{\Omega\Omega} = \frac{2}{p} - 3\gamma, \tag{91}$$

directly from the expression for M (83). Equation (90) is solved by

$$\lambda_2 = E_1\,, \quad \lambda_3 = \frac{2}{p} - 2. \tag{92}$$

The value of λ_2 violates the conditions (86), however $\lambda_3 < 0$ requires $p > 1$, which is a stronger constraint than $E_1 < 0$.

For the matter-scaling solution equation (90) gives

$$\lambda_{1,2} = \frac{3}{4}(2-\gamma)\Big[-1 \pm \sqrt{1 - 8\gamma\frac{1-3\gamma\sum_{ab}[A^{-1}]_{ab}}{2-\gamma}}\Big]\,, \quad \lambda_3 = 3\gamma - 2\,, \quad \lambda_4 = M_{\Omega\Omega}\,. \quad (93)$$

Eigenvalue λ_4 violates the conditions (86). However, the condition $\lambda_4 < 0$ is equivalent to $\lambda_3 < 0$.

In both cases the eigenvalues $\lambda_{1,2}$ are also found in section 4.2 and the eigenvalue λ_3 is additional. The condition $\lambda_3 < 0$ implies $Q < 0 \Leftrightarrow \ddot{a} > 0$. That $Q < 0$ has to be interpreted as stability against curvature perturbations is clear from equation (82). It follows that the inflationary critical points are stable against curvature perturbations. The matter-scaling solution is never inflationary since the fluid respects the strong energy condition.

Stability of critical points with $k \neq 0$

For the proper critical points with $k \neq 0$ we have $Q = \Omega = 0$ from which it follows $Y = 2X^2 = \frac{4}{3}\sum_{ab} A^{-1}_{ab}$. The curvature can be found by computing $f(X, Y, \Omega) = 2\sum_{ab} A^{-1}_{ab} - 1$.

The solutions to equation (90) are

$$\lambda_{1,2} = -1 \pm \sqrt{8\sum_{ab} A^{-1}_{ab} - 3}. \quad (94)$$

Since $\Omega = 0$, $\lambda_3 = 2 - 3\gamma$ is an eigenvalue. Stability requires that

$$\gamma > \frac{2}{3}\,, \quad \sum_{ab} A^{-1}_{ab} < \frac{1}{2}. \quad (95)$$

Hence the only stable fixed points are the ones that have $k = -1$. These are the curvature-scaling solutions of reference [17]. By a curvature-scaling solution is meant that the energy density that effectively describes the effects of spatial curvature is a constant fraction of the total energy density.

If one puts all Y_a equal to zero, then all X_i are zero and the critical point is never an attractor and represents a Milne patch of Minkowski space-time.

6 Conclusion

In this paper we study the stability of critical points in models with multiple exponential potentials and multiple fields. The critical points are separated in proper and nonproper critical points. The proper critical points are exact solutions to the equations of motion and the nonproper solutions are asymptotic descriptions of solutions. A stability analysis is carried out for models that belong to the class of generalized assisted inflation. In that class a critical point with $k = 0$ is either a kinetic-dominated solution, a fluid-dominated solution, a matter-scaling solution or a scalar-dominated solution. The critical points with $k \neq 0$ are either curvature-scaling solutions or Milne universes.

For critical points with $k \neq 0$, the curvature-scaling solution with $k = -1$ is the only stable solution.

For critical points with $k = 0$, the stability analysis shows that the fluid perturbations, curvature perturbations and scalar field perturbations all decouple from each other. The existence of a matter-scaling solution implies that there exists a scalar-dominated solution (when $p_\phi > 1/3$). Only one of the two can be stable against fluid perturbations and that is the one with the fastest powerlaw. This implies that inflationary scalar-dominated solutions are stable against fluid perturbations since we assume that the barotropic fluid respects the strong energy condition. Only the inflationary scalar-dominated solutions are stable against curvature perturbations.

Proper critical points are stable against scalar field perturbations in assisted inflation when all terms in the potential are positive. If some terms in the potential are negative there is no attractor. Since proper critical points have a higher powerlaw than the nonproper solutions, the critical point with the fastest expansion is the unique critical point that is stable against scalar field and fluid perturbations. Furthermore, if the critical point is inflationary it is stable against curvature perturbations. Therefore in assisted inflation a fast expansion is dynamically favorable.

For generalized assisted inflation the stability is more complicated. There are three main differences with assisted inflation. Firstly, nonproper critical points can be stable. Secondly, potentials with negative Λ_a can have stable (nonproper) critical points, as long as the truncation of the negative terms can be done in a stable manner. Thirdly, there can exist unstable critical points with a higher powerlaw than the stable critical points. In that case the fastest expansion is not dynamically the most favorable.

Despite the name generalized assisted inflation our models are more suited for present-day acceleration than early-universe inflation. Multiple exponential potentials either give rise to transient acceleration with too little e-foldings for early universe inflation or there exists an inflationary attractor without exit.

Regarding present-day acceleration the shortcomings of our model are different. To avoid the cosmic coincidence problem, the period where the universe enters a stage of acceleration and where the energy density of the barotropic fluid is comparable to that of the scalar fields has to be long enough. In our model such a period arises when the universe moves from an unstable matter-scaling solution towards the stable inflationary attractor [26]. During that transition there is a period with acceleration and comparable energy densities. But to make that period long enough one has to tune initial conditions. According to [27] this tuning becomes worse for multiple fields. A way to avoid this problem is by allowing interaction between the barotropic fluid and the scalar fields. Then there arises the possibility of inflationary attractors with nonzero energy density for the barotropic fluid [28].

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A Properties of the A-matrix

Consider R linearly independent vectors α_a. Their dual vectors β^b, that is, $\langle \alpha_a \, , \, \beta^b \rangle = \delta_a^b$ are uniquely determined when we demand that they all lie in $\mathrm{Span}\{\alpha_a, \ a = 1 \dots R\}$. Then we have that

$$[A^{-1}]_{ab} = \langle \beta^a \, , \, \beta^b \rangle. \tag{96}$$

The following $R-1$ vectors

$$\bar{\beta}^a = \beta^a - \frac{\langle \beta^R \, , \, \beta^a \rangle}{\langle \beta^R \, , \, \beta^R \rangle} \beta_R \quad \text{with} \quad a = 1, \dots, R-1 \, , \tag{97}$$

lie in $\mathrm{Span}\{\alpha_a, \ a = 1 \dots R-1\}$ and are such that

$$[A(R)^{-1}]_{ab} = \langle \bar{\beta}^a \, , \, \bar{\beta}^b \rangle. \tag{98}$$

Proof of the lemma

The sum of all matrix elements of A^{-1} is given by

$$\sum_{ab} [A^{-1}]_{ab} = \sum_{ab<R} \langle \beta^a \, , \, \beta^b \rangle + 2 \sum_{a<R} \langle \beta^a \, , \, \beta^R \rangle + \langle \beta^R \, , \, \beta^R \rangle \, . \tag{99}$$

If we rewrite the β's in terms of the $\bar{\beta}$'s via (97) and use that $\langle \bar{\beta}^a \, , \, \beta^R \rangle = 0$, we find

$$\begin{aligned} \sum_{ab} [A^{-1}]_{ab} = \sum_{ab<R} \langle \bar{\beta}^a \, , \, \bar{\beta}^b \rangle + \frac{1}{\langle \beta^R \, , \, \beta^R \rangle} \Big\{ \sum_{ab<R} \langle \beta^a \, , \, \beta^R \rangle \langle \beta^b \, , \, \beta^R \rangle + \\ + \sum_{a<R} \langle \beta^a \, , \, \beta^R \rangle \langle \beta^R \, , \, \beta^R \rangle + \langle \beta^R \, , \, \beta^R \rangle \langle \beta^R \, , \, \beta^R \rangle \Big\} \, . \end{aligned} \tag{100}$$

This can be rewritten as

$$\sum_{ab} [A^{-1}]_{ab} = \sum_{ab<R} [A(1)^{-1}]_{ab} + \frac{1}{\langle \beta^R \, , \, \beta^R \rangle} \Big[\sum_{a=1}^{R} \langle \beta^R \, , \, \beta^a \rangle \Big]^2 , \tag{101}$$

which proves (27) since the second term on the RHS is positive.

Stability of truncations

The condition for stability of truncating Y_1 is given by (41). If we substitute

$$X_i = \frac{1}{p}\sqrt{\tfrac{2}{3}}\sum_{ab}\alpha_{ai}[A(1)^{-1}]_{ab}\,, \tag{102}$$

in (41) the stability condition is given by

$$\sum_{a,b>1} A_{1a}\langle\bar{\beta}_a\,,\,\bar{\beta}_b\rangle > 1\,. \tag{103}$$

If we rewrite the $\bar{\beta}$ in term of the β in (104) and then use the expressions (96,98) we get

$$\sum_{a,b=1}^{M} A_{1a}\Big([A^{-1}]_{ab} - [A^{-1}]_{a1}[A^{-1}_{1b}]\Big) > 1\,. \tag{104}$$

From which (42) follows.

Using (42) one proofs the stability conditions (43) for truncating multiple Y_a in an analogues way.

Using the expressions (96,98) one can proof with the same techniques as above that if

$$\sum_{a=1}^{M}[A^{-1}]_{ab} > 0 \ \text{ for all b}\,, \tag{105}$$

then (43) can never be satisfied.

B Eigenvalues of $\partial_i\partial_j W$

Here we will proof that $\partial_i\partial_j W\mid_{\phi^*}$ has some negative values if some of the Λ_a are negative.

We use ϕ^* to denote the value of the $M-1$ scalar fields at the point where all first derivatives are zero, $\partial_i W\mid_{\phi^*}= 0$. For simplicity of notation we define the numbers $\eta_a = \Lambda_a \exp[-\kappa\langle\alpha_a,\phi^*\rangle]$ and the $M\times M$-matrix η

$$\eta = \mathrm{diag}(\eta_1,\eta_2,\ldots,\eta_M)\,. \tag{106}$$

Similarly we define the vector $\vec{\eta}$ as the vector with M components η_a.

In this notation vanishing of the first derivatives implies the following $M-1$ equations

$$\vec{\alpha_i}\cdot\vec{\eta} = 0\,, \tag{107}$$

where we indicated the $M-1$ columns of the matrix α as $\vec{\alpha}_i$. We further assume that the critical point we have in mind has $p > 1/3$ which implies that $\sum_a \eta_a > 0$.

If the eigenvalues of the $(M-1)\times(M-1)$-matrix given by

$$\partial_i\partial_j W\mid_{\phi^*}=[\alpha^T\eta\alpha]_{ij}=\sum_{a=1}^{M}\alpha_{ai}\alpha_{aj}\eta_a \tag{108}$$

are positive ϕ^* is a stable minimum otherwise it is a saddle or a maximum. If all $\eta_a>0$ it is straightforward to check that all eigenvalues are positive.

Assume, after possible renumbering, that η_1 is negative, then define the $M\times M$-matrix β as the matrix α with an extra column at the end with all entries of that column equal to 1

$$\beta=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \dots & 1\\ \alpha_{21} & \alpha_{22} & \dots & 1\\ \dots & \dots & \dots & 1\\ \alpha_{M1} & \alpha_{M2} & \dots & 1\end{pmatrix}. \tag{109}$$

The matrix α has maximal rank and therefore all columns are linearly independent. The same holds for the matrix β. The reason is that the columns of α are all perpendicular to $\vec{\eta}$, whereas the last column with all components equal to 1 has the following innerproduct with $\vec{\eta}$

$$\langle(1,\dots,1),\vec{\eta}\rangle=\sum_i\eta_i>0\,. \tag{110}$$

Hence $(1,\dots,1)$ must have a component in the $\vec{\eta}$-direction and it cannot be a linear combination of the columns of α. This means that β has maximal rank and since it is square we can define its inverse β^{-1}.

If we calculate the entries of $M\times M$-matrix $\beta^T\eta\beta$ using the identity (107) we find that it takes the form

$$\beta^T\eta\beta=\begin{pmatrix}\alpha^T\eta\alpha & 0\\ 0 & \sum_a\eta_a\end{pmatrix}. \tag{111}$$

We notice that the spectrum of $\beta^T\eta\beta$ equals the spectrum of $\alpha^T\eta\alpha$ plus the positive eigenvalue $\lambda=\sum_a\eta_a$. If $\beta^T\eta\beta$ has a negative eigenvalue so has $\alpha^T\eta\alpha$. Therefore it is enough to prove that there exists a vector ψ such that $\langle\psi|\beta^T\eta\beta|\psi\rangle<0\,.$ It is easily checked that

$$\psi=\beta^{-1}\begin{pmatrix}1\\0\\\dots\\0\end{pmatrix} \tag{112}$$

is such a vector.

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